The core of hedonic partitioning games

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Abstract. In this note we introduce the class of partitioning hedonic games, which extends, to the field of hedonic games, a very interesting class of games already studied in games with and without transferable utility. We show that the same condition that guarantees the existence of non-empty core for any characteristic function, already proven when a utility is present, also guarantees the existence of non-empty core, for any preference profile, when a utility is absent.

1 Introduction

In their seminal paper, Kaneko and Wooders [7] introduce the class of partitioning games as a way to capture the fact that "... in an n-cooperative game could not be equally easy to form every coalition". In a partitioning game there is a subset of coalitions which plays such an essential role that determines the behavior of all the other coalitions. This characteristic is shared for several of the games studied in the literature such as the marriage game (Gale and Shapley [6]), the bridge game (Shubik [11]), the assignment game (Shapley and Shubik [10]) and the m-sided assignment game (Quint [9]) among others. Kaneko and Wooders [7] present a transferable utility version for the partitioning games and a non-transferable utility version as well. Their work focuses on the non-emptiness of the core of the games. As a key contribution, they provide a list of necessary and sufficient conditions to characterize a class of restricted families of coalitions under which any possible induced partitioning game has non-empty core. The first class of games studied in the literature having a restricted family of coalitions with this property is that of the assignment games (Shapley and Shubik [10]). On the other hand, while the essential coalitions in the marriage game satisfy the conditions stated by Kaneko and Wooders [7], none of the versions of the partitioning game presented by them suit well to deal with the existence of stable matchings. In this note we introduce a hedonic version of the partitioning game, and show that it exhibits core-properties which are similar to those already proved for partitioning games with and without transferable utility. Our approach can be used to provide another non-constructive proof for the existence of stable matching in the marriage game ([4]), being that of Sotomayor [12] the first we have seen published.

In the next section we introduce hedonic partitioning games as stated by Banerjee et al. [1] and Bogomolnaia and Jackson [2] and also our version of the games where some restrictions on the family of admissible coalitions is imposed. For this latter class of games we state two balancedness conditions resembling those of ordinal balancedness of Bogolmolnaia and Jackson [2] and pivotal balancedness stated by Iehlé [5] for hedonic games without restrictions on the set of coalition that can be formed. Following those authors, we show that both conditions are sufficient to guarantee the existence of core partitions, while pivotal balancedness is also a necessary condition. In Section 3 we introduce the partitioning hedonic game, which always has a family of essential coalitions associated, and prove the main results of the note which states that every partitioning game has non-empty core if the family of admissible coalitions satisfies either condition *ii*) or the equivalent condition *iii*) of Theorem 2.7 of Kaneko and Wooders [7]. We close with some concluding remarks.

2 Hedonic partitioning games

We start with a finite set $N = \{1, ..., n\}$ whose elements are going to be called the players, while a subset of them will be a coalition. The family of nonempty coalitions will be denoted by $\mathcal{N}$. Given any family $\mathcal{B}$ of coalitions, and a player $i \in N$, let us denote by $\mathcal{B}(i)$ the subfamily of coalitions in $\mathcal{B}$ containing player i. A hedonic game is a pair $(N, \succeq)$, where $\succeq = (\succeq_i)_{i \in N}$ is a preference profile with $\succeq_i$ being a reflexive, complete and transitive binary relation on $\mathcal{N}(i)$ for each $i \in N$. For each $i \in N, \succ_i$ will stand for the strict preference relation related to $\succeq_i$ ($S \succ_i T$ iff $S \succeq_i T$ but not $T \succeq_i S$). $\mathcal{P}(N)$ will denote the family of partitions of N. Given $\pi = \{\pi_1, ..., \pi_p\} \in \mathcal{P}(N)$ and $i \in N, \pi(i)$ will denote the unique set in π containing player i.

Given a hedonic game $(N, \succeq)$ and $\pi \in \mathcal{P}(N)$, we say that $T \in \mathcal{N}$ blocks π if for each $i \in T, T \succ_i \pi(i)$. The core $C(N, \succeq)$ of $(N, \succeq)$ is the set of partitions blocked by no coalition.

A a non-empty collection $\mathcal{A} \subseteq \mathcal{N}$, such that $\{i\} \in \mathcal{A}$ for all $i \in N$ is called basic (Kaneko and Wooders [7]), Quint [9]) or effective (Le Breton et al. [8]). Coalitions in such a family will be the essential coalitions in a partitioning game. But first, we introduce another interesting class of hedonic games. Given an basic family $\mathcal{A}$ and a preference profile $\hat{\succeq} = (\hat{\succeq}_i)$ restricted to $\mathcal{A}$, namely, where for each $i \in N, \hat{\succeq}_i$ is a reflexive, complete and transitive binary relation on $\mathcal{A}(i), (N, \hat{\succeq}; \mathcal{A})$ will be referred as a hedonic game with $\mathcal{A}$ as its family of admissible coalitions (Cesco [3]). Let $S \in \mathcal{N}$. An $\mathcal{A}$-partition of S is a partition π^S of S such that all the members of π^S belong to $\mathcal{A}$. Let $\pi(S) = \{T \in \mathcal{A}$: there exists π^S such that $T \in \pi^S\}$. Clearly $\pi(S) = \{T \in \mathcal{A} : T \subseteq S\}$.

A family of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called balanced if there exists a set of positive real numbers $(\lambda_S)_{S \in \mathcal{B}}$ satisfying $\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \lambda_S = 1$, for all $i \in N$. The numbers $\lambda_S, S \in \mathcal{B}$ are called the balancing weights for $\mathcal{B}$. $\mathcal{B}$ is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balanced weights is unique. Let us call the family of basic coalitions $\mathcal{A}$ partitionable (Kaneko and Wooders [7]) if the only minimal subfamilies that it contains are partitions.

A family $\mathcal{I} = (\mathcal{I}(A))_{A \in \mathcal{A}}$ is called an $\mathcal{A}$-distribution, or simply a distribution (Iehlé [5]) if, for each non-empty coalition $A \in \mathcal{A}$, $\phi \neq \mathcal{I}(A) \subseteq A$. Given a distribution $\mathcal{I}$, a family $\mathcal{B} \subseteq \mathcal{A}$ is $\mathcal{I}$-balanced if the family $(\mathcal{I}(B))_{B \in \mathcal{B}}$ is balanced.

Definition 1. $(N, \hat{\succeq}; \mathcal{A})$ *is ordinally balanced if for each balanced family* $\mathcal{B} \subseteq \mathcal{A}$ *there exists an* $\mathcal{A}-$*partition* π *of* N *such that, for each* $i \in N, \pi(i) \hat{\succeq}_i B$ *for some* $B \in \mathcal{B}(i)$.

Ordinal balancedness is first stated by Bogomolnaia and Jackson [2] for the case $\mathcal{A} = \mathcal{N}$.

Definition 2. $(N, \hat{\succeq}; \mathcal{A})$ *is pivotally balanced with respect to an* $\mathcal{A}$*-distribution* $\mathcal{I}$, *if for each* $\mathcal{I}$*-balanced family* $\mathcal{B}$, *there exists an* $\mathcal{A}-$*partition* π *of* N *such that, for each* $i \in N, \pi(i) \hat{\succeq}_i B$ *for some* $B \in \mathcal{B}(i)$. *The game is pivotally balanced if it is pivotally balanced with respect to some distribution* $\mathcal{I}$.

This general concept of balancedness is studied by Iehlé [5] for the case $\mathcal{A} = \mathcal{N}$.

The first part of the following theorem is a sufficient condition for the existence of core-partitions for hedonic games with coalitional restrictions which parallels the first part of Theorem 1 of Bogomolnaia and Jackson [2], while the second part parallels the characterization given in Theorem 3 of Iehlé [5], and whose proofs are carried out in a similar way.

Theorem 1. *Let* $(N, \hat{\succeq}; \mathcal{A})$ *be a hedonic game with* $\mathcal{A}$ *as its family of admissible coalitions.*

Theorem 2. *a) If the game is ordinally balanced, and has strict individual preferences, then* $C(N, \hat{\succeq}; \mathcal{A})$ *is non-empty.*

Theorem 3. *b)* $C(N, \hat{\succeq}; \mathcal{A})$ *is non-empty if and only if the game is pivotally balanced.*

Remark 1. Ordinal balancedness implies pivotal balancedness with respect to the distribution $\mathcal{I} = (\chi_A)_{A \in \mathcal{A}}$, where χ_A stands for the indicator vector of the coalition A.

3 Partitioning hedonic games

The idea behind a partitioning game is that all the development of the game depends on a restricted family of coalitions $\mathcal{A}$. To capture this characteristic, the behavior of any coalition S outside the family $\mathcal{A}$ is determined by all of its $\mathcal{A}$-partitions in such a way that none of these arrangements are better off, for any player $i \in S$, than any sub-coalition in $\mathcal{A}$ of S containing i. To define a partitioning hedonic game we start with a hedonic game $(N, \hat{\succeq}; \mathcal{A})$ with a restricted family $\mathcal{A}$ of admissible coalitions which we will call the germs of the partitioning hedonic game.

Definition 3. *A partitioning hedonic game is a hedonic game* $(N, \succeq)$ *for which there exists a germs* $(N, \hat{\succeq}; \mathcal{A})$ *such that, for all* $i \in N, \succeq_i$ *is defined as:*

$$S \succeq_i T \text{ if and only if } \gamma(S;i)\hat{\succeq}_i\gamma(T;i),$$

where, for all $S \in \mathcal{N}(i)$,

$$\gamma(S;i) = S \text{ if } S \in \mathcal{A},$$

and, if $S \notin \mathcal{A}$

$$\gamma(S;i) = S^*,$$

with $S^* \in \pi(S) \cap \mathcal{A}(i)$ *satisfying* $T\hat{\succeq}_i S^*$ *for all* $T \in \pi(S) \cap \mathcal{A}(i)$.

Let us use $(N, \succeq \mid \mathcal{A})$ to denote a hedonic partitioning game with $\mathcal{A}$ as its family of basic coalitions.

Theorem 4. *Let a partitionable basic family of coalitions* $\mathcal{A}$ *be given. Then, every partitioning hedonic game* $(N, \succeq \mid \mathcal{A})$ *has non-empty core.*

Proof. We first note that the germ $(N, \hat{\succeq}; \mathcal{A})$ of the partitioning game is ordinally balanced, and thus, it has a non-empty core. To see this, let $\mathcal{B}$ be a balanced family of coalitions and because $\mathcal{A}$ is partitionable, $\mathcal{B}$ contains a partition π. Then, since for each $i \in N$ it holds that $\pi(i)\hat{\succeq}_i\pi(i)$, we conclude that the game is ordinally balanced. We point out that being the individual preferences in the game not necessarily strict, the non-emptiness of the core is guaranteed by part b) of Theorem 1 rather than by part a).

Now, let $\pi \in C(N, \hat{\succeq}; \mathcal{A})$. We will show that $\pi \in C(N, \succeq \mid \mathcal{A})$ too. Clearly there is no $S \in \mathcal{A}$ objecting π. And if $S \notin \mathcal{A}$, for each $i \in S$ it holds that

$$\pi(i)\hat{\succeq}_i S,$$

so S can not object π either. This completes the proof.

3.1 Concluding remarks

In this note we have extended the notion of partitioning game, in a natural way, to the framework of hedonic games, showing that this new class of games exhibits the same important feature of having non-empty core, for any preference profile, under the same condition that Kaneko and Wooders [7] used to guarantee that any partitioning game has non-empty core in the framework of games where a utility, transferable or not, is present. We would like to mention that our result can be used to provide another non-constructive proof (Cesco [4]) of the existence of stable matchings in the marriage model of Gale and Shapley [6], with a different approach to that used in Sotomayor [12].

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